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## Chiral Fermions, Gravity and GUTs<sup>@</sup>

Lay Nam Chang\* and Chopin Soo<sup>†</sup>

\*Institute for High Energy Physics and Dept. of Physics  
Virginia Polytechnic Institute and State University  
Blacksburg, VA 24061-0435

<sup>†</sup>Center for Gravitational Physics and Geometry  
Department of Physics  
Pennsylvania State University  
University Park, PA 16802-6300

### ABSTRACT

We discuss a global anomaly associated with the coupling of chiral Weyl fermions to gravity. The Standard Model based upon  $SU(3) \times SU(2) \times U(1)$  which has 15 fermions per generation is shown to be inconsistent if all background spin manifolds with signature invariant  $\tau = 8k$  are allowed. Similarly, GUTs based on odd number of fermions are inconsistent. Consistency can be achieved by adding an extra Weyl fermion which needs to couple only to gravity. For arbitrary  $\tau$ 's, generalized spin structures are needed, and the global anomaly cancellation requires that the net index of the total Dirac operator with spin and internal gauge connections be even. As a result GUTs with fundamental multiplets which contain multiples of 16 Weyls per generation are selected. The simplest consistent GUT is the  $SO(10)$  model with a multiplet of 16 Weyls per generation. The combined gravity and internal symmetry gauge group of the theory is then  $[Spin(3,1) \times Spin(10)]/Z_2$ . Physical implications of these results are commented on.

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\*laynam@lotus.cc.vt.edu; <sup>†</sup> soo@phys.psu.edu

Witten[1] uncovered a global anomaly which states that in 4 dimensions, a gauge theory with an odd number of chiral fermion doublets coupled to  $SU(2)$  gauge fields is inconsistent. In the Standard Model based upon  $SU_C(3) \times SU_W(2) \times U_Y(1)$ , the number of left-handed fermion doublets coupled to  $SU_W(2)$  turns out to be reassuringly even (4 per generation). However, when the gravitational field is taken into account, there is an *additional*  $SU(2)$  gauge group which comes from the rotation subgroup of the local Lorentz gauge group of gravity. An immediate question is if additional constraints need to be imposed on models of Particle Physics when interactions with the gravitational fields are included[2]. In particular, these consistency conditions can arise from treating Weyl fermions as quantum fields in gravitational backgrounds of arbitrary topologies. Witten's arguments have been elaborated on by others[3]. We review the essential points, and show how an inconsistency can arise. The relevant doublets for the case we have at hand are the two components of Weyl fermions.

To begin with, consider a suitable Wick rotation of the background spacetime into a Riemannian manifold, or the more general setting of matter coupled to gravity in Euclidean Quantum Gravity. The fermions can be expanded in terms of the complete set of the eigenfunctions,  $\{X_n\}$ , of the hermitean Dirac operator [4]. The generating functional,  $Z_W[\omega]$ , for a left-handed 2-component Weyl doublet coupled to the gravitational spin connection,  $\omega$ , can be defined to be the square-root of the generating functional of a bispinor. (More precisely, one can think of each Weyl doublet coupled to gravity as a Majorana bispinor). We then have

$$Z_W[\omega] = Z_{Dirac}^{\frac{1}{2}}[\omega] = \{\det(i\gamma^\mu D_\mu^\omega)\}^{\frac{1}{2}} \quad (1)$$

Consider next a chiral transformation by  $\pi$  which maps each 2-component left-handed Weyl fermion  $\Psi_L(x) \mapsto \exp(i\pi\gamma_5)\Psi_L(x) = \exp(-i\pi)\Psi_L(x) = -\Psi_L(x)$ . Obviously such a map is a symmetry of the action. However, the measure is not necessarily invariant [4] under such a chiral transformation because of the Adler-Bell-Jackiw anomaly[5]. Instead, each left-handed fermion measure transforms as

$$d\mu \mapsto d\mu \exp(-i\pi \int_M d^4x \det(e) \sum_n X_n^\dagger(x) \gamma_5 X_n(x)) \quad (2)$$

The expression  $\int_M d^4x \sum_n \det(e) X_n^\dagger(x) \gamma_5 X_n(x)$ , (which needs to be regularized) is formally equal to  $(n_+ - n_-)$ , where  $n_\pm$  are the number of normalizable positive and negative chirality zero modes of the Dirac operator. Upon

regularization, the expression works out to be[4]

$$\begin{aligned}
\sum_n \det(e) X_n^\dagger \gamma_5 X_n &\equiv \lim_{\mathcal{M} \rightarrow \infty} \sum_n \det(e) X_n^\dagger(x) \gamma_5 e^{-(\lambda_n/\mathcal{M})^2} X_n(x) \\
&= \lim_{\mathcal{M} \rightarrow \infty} \lim_{x' \rightarrow x} \text{Tr} \gamma_5 \det(e) e^{-(i\gamma^\mu D_\mu/\mathcal{M})^2} \sum_n X_n(x) X_n^\dagger(x') \\
&= -\frac{1}{384\pi^2} R_{\mu\nu\sigma\tau} * R^{\mu\nu\sigma\tau}
\end{aligned} \tag{3}$$

As a result,  $(n_+ - n_-) = \sum_n \int_M d^4x \det(e) X_n^\dagger(x) \gamma_5 X_n(x) = -\tau/8$ , where  $\tau = (1/48\pi^2) \int_M R_{\mu\nu\sigma\tau} * R^{\mu\nu\sigma\tau}$ , is the signature invariant of the four-manifold  $M$ . (In this discussion, we shall restrict our attention to compact orientable 4-manifolds without boundary.) If there are altogether  $N$  left-handed Weyl fermions in the theory, the total measure changes by  $\exp(iN\pi\tau/8)$ .

But, the chiral transformation of  $-1$  on the fermions can also be considered to be an ordinary  $SU(2)$  rotation of  $2\pi$ . Since  $SU(2)$  is a safe group with no perturbative chiral anomalies (there are no Lorentz anomalies in 4 dimensions), the measure must be invariant under all  $SU(2)$  transformations. Thus an *inconsistency* arises unless the phase factor,  $\exp(iN\pi\tau/8)$ , is always unity.

It is known that for consistency of parallel transport of spinors for topological four-manifolds,  $\tau$  must be a multiple of 8 for spin structures to exist. If in quantum gravity, or in semiclassical quantum field theories, one allows only for four-manifolds with  $\tau = 16k, k \in \mathbb{Z}$ ; then under a chiral rotation of  $\pi$  the measure is invariant regardless of  $N$ . Otherwise, consistency with the global anomaly requires that  $N$  must be even if all spin manifolds with  $\tau = 8k$  are permitted. If one counts the number of left-handed Weyl fermions in the Standard Model, one finds that the number is 15 per generation, giving a total of 45 for 3 generations. This comes about because each bispinor is coupled twice to the spin connection while each Weyl spinor is coupled once (e.g. for the first generation, the number is 2 for each electron and each up or down quark of a particular color, and 1 for each left-handed neutrino.) Thus even if one restricts to  $\tau = 8k$ , the global anomaly with respect to  $SU(2)$  rotations implies that there should be additional particle(s). For example, there could be a partner for each neutrino, making  $N$  to be 16 per generation, or a partner for just the  $\tau$ -neutrino, or even four generations. As a result, even with  $\tau = 8k$ , grand unification schemes based upon groups such as  $SU(5)$  and odd number of Weyl fermions would be inconsistent when coupled to gravity.

It is quite likely that in quantum gravity the allowed manifolds should extend over a wider class than those which admit classical spin structures[6]. The considerations outlined above suggest that in order to do this, we would have to allow for couplings via gauge fields among the various fermions, which is what occurs in Grand Unified Theories (GUTs). The reason can be stated as follows. When  $\tau$  is not a multiple of 8, it is not possible to lift the  $SO(4)$  bundle to its double cover  $Spin(4)$  bundle since the second Stiefel-Whitney class is non-trivial. However, given a general grand unification simply-connected gauge group  $G$  with a  $Z_2 = \{e, c\}$  in its center, it is possible to construct *generalized spin structures* with gauge group  $Spin_G(4) = \{Spin(4) \times G\}/Z_2$  where the  $Z_2$  equivalence relation is defined by  $(x, g) \equiv (-x, cg)$  for all  $(x, g) \in Spin(4) \times G$ [6]. Note that  $Spin_G(4)$  is the double cover of  $SO(4) \times (G/Z_2)$ . The parallel transport of fermions then does not give rise to any inconsistency.

Now, in a scenario where all of the fermions are coupled to one another, the index theorem should be applied only to the trace current involving the sum over all the fermion fields. The resultant restriction on  $N$  is the condition that the index for the *total* Dirac operator with  $Spin_G(4)$  connections,  $N_+ - N_-$ , is even. The index is given by

$$N_+ - N_- = -N\tau/8 - 1/(8\pi^2) \int_M Tr(F \wedge F) \quad (4)$$

where  $F$  is the curvature of the GUT group and  $N$  is the total number of Weyl fermions in the GUT multiplet. The net conclusion is therefore that inclusion of manifolds with arbitrary  $\tau$ 's is possible, provided we allow for generalized spin structures, with a total of  $16k$  fermions and a GUT-group with always even instanton numbers, in order to ensure that the global anomaly is absent.

In the event that the structures are defined by simple GUT groups, the preeminent choice would be  $SO(10)$ [7]. It is easy to check that the 16 Weyl fermions in the  $SO(10) = Spin(10)/Z_2$  GUT indeed belong to a 16-dim. representation of  $Spin(10)$ , and satisfy the generalized spin structure equivalence relation for  $\{Spin(4) \times Spin(10)\}/Z_2$ . Also,  $1/(8\pi^2) \int_M Tr(F \wedge F)$  is always even. It is worth emphasizing that the generalized spin structure implies an additional "isospin-spin" relation in that fermions must belong to  $Spin(10)$  representations, while bosons must belong to  $SO(10)$  representations of the GUT. This has implications for spontaneous symmetry breaking via fundamental bosonic Higgs, which cannot belong to the spinorial representations of  $SO(10)$  if one allows for arbitrary  $\tau$ 's. More generally, the

presence of the extra particle(s) implied by global anomaly considerations can generate masses for neutrinos, thereby giving rise to neutrino oscillations, and also play a significant role in the cosmological issue of “dark matter”.

We have presented our arguments in terms of conventional spin connection couplings for definiteness. We would like to end by emphasizing that our results would also obtain [2] in the setting of Weyl fermions coupled to Ashtekar-Sen connections in the Ashtekar formulation of gravity [8].

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